Research Practicum:
Experimental Violation of Bell's Inequality

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0 Introduction

This document is intended for you to use as a 'cheat sheet' during performing the actual experiment, i.e. it contains all informations (formulas, etc.) in a compact form. However, it does not enable you to comprehend the physics that underly the experiment, and it is not intended to do so! Please refer to the given references to understand the foundations! Also, make sure you have read all the supplied information (Blackboard!) before starting the experiment.

Section 2 contains some questions with which you can check whether you understand the basic principles.

Very importantly, section 4 contains a list of measurements which you are going to perform and evaluate.

The appendix contains some more technical information on how to do certain things and the underlying basics.

No worries, you will not be left alone in the lab with this, there will always be one supervisor which you can ask for assistance. However, it is probably not a bad idea if you have all this information at hand prior to performing the experiment so you can already think about how everything works in advance. Good luck, and, very important, have fun!

1 Conventions

In the following, we will stick to notation and formalism as outlined below. When talking about angles, this is always relative to some universal value Θ_{ref} that is used as a reference. We omit this by setting $\Theta_{\mathrm{ref}} = 0$. This could, for instance, correspond to the case that the $|\mathrm{H}\rangle$ ($|\mathrm{V}\rangle$) polarization is really lying horizontally (vertically) in the laboratory frame, but note that this is not necessarily the case. When talking about the rotation of a basis at either Alice's or Bob's side, given by an angle Θ_{A} or Θ_{B}, this is always relative to some zero point, and means, of course, that both $|\mathrm{H}\rangle$ and $|\mathrm{V}\rangle$ polarizations are rotated by this angle. Since both detector sides, A and B, require two different basis rotation angles to measure the violation of Bell's inequality, we introduce the second angles as Θ'_{A} and Θ'_{B}.

When we talk about photon pair states, we usually write them in a form similar to

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|\mathrm{H}\rangle|\mathrm{V}\rangle + \mathrm{e}^{\mathrm{i}\phi}\, |\mathrm{V}\rangle|\mathrm{H}\rangle \right].$$

For both products in this state, we'll assume that the first basis state corresponds to the photon that is sent to Alice, the latter one to Bob. Product states we may also abbreviate like

$$|\mathrm{H}\rangle|\mathrm{V}\rangle \equiv |\mathrm{HV}\rangle\,.$$

To show that Bell's inequality does not hold in reality, we have to measure [1]

$$|S| > 2.$$

The value S is a function given by the expression

$$S = E(\Theta_\mathrm{A}, \Theta_\mathrm{B}) + E(\Theta'_\mathrm{A}, \Theta_\mathrm{B}) - E(\Theta_\mathrm{A}, \Theta'_\mathrm{B}) + E(\Theta'_\mathrm{A}, \Theta'_\mathrm{B}),$$

where in turn the function $E(\theta, \varphi)$ is given by

$$E(\theta, \varphi) = P_\mathrm{HH}(\theta, \varphi) + P_\mathrm{VV}(\theta, \varphi) - P_\mathrm{HV}(\theta, \varphi) - P_\mathrm{VH}(\theta, \varphi).$$

P_XY denotes the probability to measure one photon with polarization $|\mathrm{X}\rangle$ at side A and the other photon of the pair with polarization $|\mathrm{Y}\rangle$ at side B, where the angle θ (φ) corresponds to the basis rotations on side A (B). Since in the lab we measure probabilities by counting for a certain time, we can write this as

$$E(\theta, \varphi) = \frac{N_\mathrm{HH}(\theta, \varphi) + N_\mathrm{VV}(\theta, \varphi) - N_\mathrm{HV}(\theta, \varphi) - N_\mathrm{VH}(\theta, \varphi)}{N_\mathrm{HH}(\theta, \varphi) + N_\mathrm{VV}(\theta, \varphi) + N_\mathrm{HV}(\theta, \varphi) + N_\mathrm{VH}(\theta, \varphi)},$$

where N is the number of counts in a given interval of time (i.e., the count rate). All arguments and indices keep their meaning.

2 Introductory Questions

In order to fully get the point of what you are doing during the experiment, it is important that you properly understand the underlying physical concepts. Before you do the experiment, make sure that you can answer the following questions. It is not so important that you calculate everything in great detail, but that you know how to get there and how to handle the principles.

2.1 Entanglement

General introductions to the concepts of entanglement can be found for instance in [1] or [2].

2.1.1 Photon pairs I

Consider first a light source that emits photon pairs in the following fashion: One photon is always sent to A, the other one always to B. One of the photons is always polarized as $|\mathrm{H}\rangle$, the other one as $|\mathrm{V}\rangle$ (in a fixed frame, that is – this could be the lab frame, for instance). The probability for either polarization to be sent to a destination is 50%. What are the probabilites P_XY (where both X and Y can be both H and V) with which we expect photons arriving at A and B?

Now, let's have a look at at an entangled photon pair in the form of

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|\mathrm{HV}\rangle + |\mathrm{VH}\rangle\right].$$

If we perform the exact same experiment with these photons, what are the probabilities now? Can we distinguish between the first source and the entangled case?

2.1.2 Photon pairs II

We continue the experiment with both sources, but now let's rotate Alice's as well as Bob's measurement basis by $\Theta_\mathrm{A} = \pi/4 = 45°$. What are the probabilities now?

[1] The derivation of the general Bell's inequality, be it in its original form or in the so-called CHSH form — the one that you read about at this very moment — is not exactly trivial! This is why most introductory texts do not bother to give it, since it's basically only math, and it can be hard to see the relation to real physical quantities. Therefore it's enough if you follow the basic approach given in the experiment manual or in [2].

2.2 Wave plates

Wave plates are thin slices of birefringent materials. These materials have the property that light propagation inside depends on polarization. To be more specific, the polarization component parallel to the so called 'optical axis' (here, let us assume this is the $|\mathrm{V}\rangle$ polarization) passes faster than the component perpendicular to it ($|\mathrm{H}\rangle$). It is obvious that this leads to a phase shift between these two polarization components, when light passes through the wave plate. What people make use of frequently when they want to alter the polarization properties of their light, are $\frac{\lambda}{2}$-plates, or half-wave plates (HWP), and $\frac{\lambda}{4}$-plates, or quarter-wave plates (QWP). For the first case, this means that the $|\mathrm{H}\rangle$ component of the light has been retarded by half a wave with respect to the $|\mathrm{V}\rangle$ component, indicating a phase shift of $\Delta\phi = \pi$. For the quarter-wave plate, as you will have guessed by now, the retardation is quarter a wave, indicating a phase shift of $\Delta\phi = \pi/2$.

In practice, this means that a HWP 'flips' the polarization of light on the fast axis of the crystal. It is easy to see that this leads to the fact that you can rotate the polarization like this: A rotation of an angle Θ of the wave-plate results in a 2Θ rotation of the polarization. A QWP, in contrast, can be used to obtain circularly or elliptically polarized light.

In principle, this is all you need to know to be able to use a wave plate, but for a proper understanding, you might want to consult an optics book, e.g. [3].

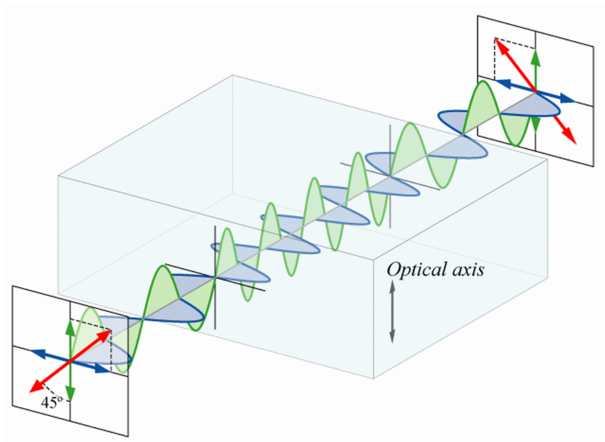

Figure 1: Working principle of a wave plate. The polarization component perpendicular to the optical axis obtains a gradually increasing phase shift while traveling through the material. In the example here, the total shift is $\pi/2$, which corresponds to rotation of twice the angle between the original polarization and the optical axis. Half the thickness of the plate would result in a smaller phase shift, so that the light after the plate could be circularly of elliptically polarized. Taken from [4].

2.2.1 Polarization rotation

Assume a photon with polarization state

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left[|\mathrm{H}\rangle + |\mathrm{V}\rangle \right],$$

i.e., its polarized at an angle of 45° with respect to the optical axis, is sent through a half-wave plate. How does its polarization change? Can you generalize this for an arbitrary angle between the polarization and the optical axis?

Now, let's turn the optical axis, i.e., the wave plate, by some angle θ. How does the polarization after the wave plate change compared to the situation before the wave plate rotation?

2.2.2 More Phases and Amplitudes

What happens if we send the previously discussed photon through a quarter-plate? What is the behavior here for an arbitrary angle?

3 Setup

Figure 2 shows a schematic of the setup that we use during the experiment. The creation of polarization-entangled photons is described in [5]. This part of the setup is treated as a 'black box' during the experiment. Both rays of entangled photons are are sent to Alice and Bob via optical fibers, where their polarization can be altered (see chapter 2.2 and appendix A.1). The light then incidents on a polarizing beam splitter (PBS), which has the effect that horizontally polarized light is passed through, whereas vertically polarized light gets deflected[2]. The counts on the four photon detectors (AH, AV, BH, BV, according to recipient and light polarization that they detect) are then analyzed via a correlation counter (a correlated count is only present if one detector on side A and one on side B give a count within a 4 ns time window).

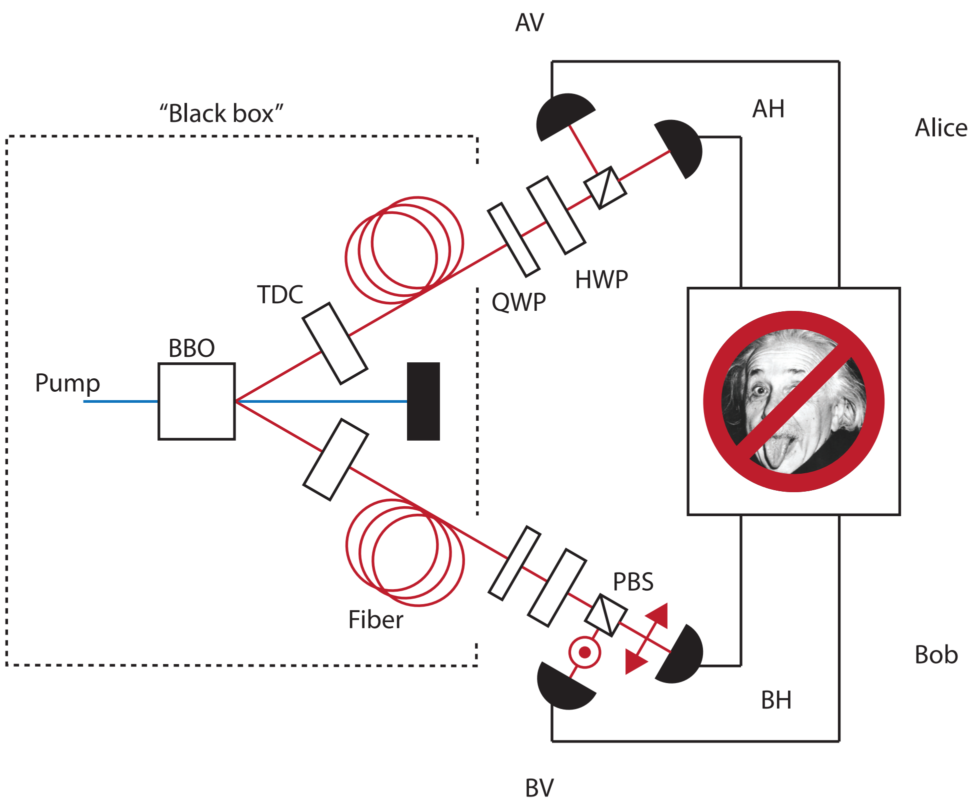

Figure 2: Schematic of the setup, top view. *Pump*: Blue pump laser; *BBO*: Crystal in which the entangled photons are created; *TDC*: Time-delay (and walk-off) compensation crystal; *QWP*: Quarter wave-plate; *HWP*: Half wave-plate; *PBS*: Polarizing beam-splitter. More details are in the accompanying text.

[2]This means, that with our setup we do not alter the measurement basis, but instead rotate the photon's polarization basis. In the literature you can find cases where a polarizer is rotated instead of the photon polarization. Any angle given in such a type of setup has then a negative sense of rotation compared to the setup we use.

4 Measurement Tasks

After performing the experiment, you are required to write and hand in a written report about what you did. It is therefore crucial that you take detailed notes about the actions you take and what you observe.

4.1 Calibration

First, use the wave plates to obtain the state $|\Psi^-\rangle$. Information on how to do this you can find in appendix A.1. Most likely you will notice that the correlations are not perfect, i.e., there are also coincidence counts that you'd not expect for a 'perfect' Bell state and experiment setup. Use this to find a value for the effective 'visibility', or 'fidelity', of the observed pair states as described in appendix B. What does this mean for the value of S we can expect in our setup?

4.2 Observation of Nonlocality

Using the state that has now been prepared, we will now see if quantum mechanical predictions that violate locality can be observed in our lab. Using the basis rotation angles that predict maximum violation,

$$\begin{aligned}
\Theta_{\mathrm{A}} &= 0, \\
\Theta'_{\mathrm{A}} &= 45°, \\
\Theta_{\mathrm{B}} &= 22.5°, \\
\Theta'_{\mathrm{B}} &= 67.5°,
\end{aligned}$$

determine the parameter S. Does it match your expectations? If not, what do you suspect are possible explanations?

4.3 Basis Dependence

We now want to see the dependence of the measurements on the basis angles. For this, we'd like to measure two correlation curves. For $\Theta_A = 0°, 45°$, rotate Θ_B in steps of $10°$ for a full round trip and determine $E(\Theta_A, \Theta_B)$ for each point.

5 Report

You are required to hand in a written report on what you did during the practicum. In this, please include a short overview over the important theoretical principles to show what measurement outcome is expected. This should be followed by the explaination and interpretation of your experimental results. For this analysis, include your measured data and do an error analysis. Explain what could be sources of errors in your opinion and how they might have influenced your measurements. Also, interpret your data and state your conclusions.

References

[1] David J. Griffiths. *Introduction to Quantum Mechanics.* Prentice Hall, 1st edition, 1994.

[2] Mark Fox. *Quantum Optics.* Oxford University Press, Oxford, 1st edition, 2006.

[3] Eugene Hecht. *Optics.* Addison-Wesley, San Francisco, 4th edition, 2002.

[4] Wave plate — Wikipedia, The Free Encyclopedia.

[5] Paul Kwiat, Klaus Mattle, Harald Weinfurter, Anton Zeilinger, Alexander Sergienko, and Yanhua Shih. New High-Intensity Source of Polarization-Entangled Photon Pairs. *Physical Review Letters*, 75(24):4337–4341, 1995.

A Bell States

A.1 Preparation

The four Bell states (the so-called ‘maximally entangled’ states that form a full basis of the Hilbert space for two entangled particles) in our case are given by

$$\left|\Psi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\mathrm{HV}\right\rangle \pm \left|\mathrm{VH}\right\rangle\right],$$
$$\left|\Phi^{\pm}\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\mathrm{HH}\right\rangle \pm \left|\mathrm{VV}\right\rangle\right].$$

However, the general state of the photon pair before entering the optical fibers that lead to the analyzation part of the setup is given by

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\mathrm{HV}\right\rangle + \mathrm{e}^{\mathrm{i}\phi}\left|\mathrm{VH}\right\rangle\right].$$

In general, the phase ϕ is adjusted by the time compensation system in the setup [5], so this is nothing during the experiment for you to worry about. In the following, we assume

$$\phi = 0.$$

By proper adjustment of the half-wave plates in front the analyzation part (i.e., polarizing beam splitters (PBS), and photon counters) we can rotate the H and V components to match the detectors' basis, where due to the setup geometry H (V) matches the horizontal (vertical) direction in the lab frame. Thus, in the detector basis, we have the state

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\mathrm{HV}\right\rangle + \left|\mathrm{VH}\right\rangle\right] \equiv \left|\Psi^{+}\right\rangle.$$

By adjusting the quarter-wave plates, a phase shift of $\phi = \pi$ between the two components of this state can be acquired[3], which gives us

$$\left|\psi\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\mathrm{HV}\right\rangle - \left|\mathrm{VH}\right\rangle\right] \equiv \left|\Psi^{-}\right\rangle.$$

To see how we can discriminate the two cases experimentally, let's rotate the polarization on both detector sides A and B by $\pi/4$ (i.e. we rotate both half-wave plates by 22.5° in the same sense). In the new photon base, the base vectors are given, expressed by the detector base states, as

$$\left|\mathrm{H}'\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\mathrm{H}\right\rangle - \left|\mathrm{V}\right\rangle\right],$$
$$\left|\mathrm{V}'\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\mathrm{H}\right\rangle + \left|\mathrm{V}\right\rangle\right].$$

Therefore, the states read now, in the detector base,

$$\begin{aligned}\left|\Psi^{+,\prime}\right\rangle &= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\left|\mathrm{H}\right\rangle - \left|\mathrm{V}\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|\mathrm{H}\right\rangle + \left|\mathrm{V}\right\rangle\right) + \frac{1}{\sqrt{2}}\left(\left|\mathrm{H}\right\rangle + \left|\mathrm{V}\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|\mathrm{H}\right\rangle - \left|\mathrm{V}\right\rangle\right)\right]\\ &= \frac{1}{\sqrt{2}}\left[\left|\mathrm{HH}\right\rangle - \left|\mathrm{VV}\right\rangle\right] \equiv \left|\Phi^{-}\right\rangle,\end{aligned}$$

and

$$\begin{aligned}\left|\Psi^{-,\prime}\right\rangle &= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(\left|\mathrm{H}\right\rangle - \left|\mathrm{V}\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|\mathrm{H}\right\rangle + \left|\mathrm{V}\right\rangle\right) - \frac{1}{\sqrt{2}}\left(\left|\mathrm{H}\right\rangle + \left|\mathrm{V}\right\rangle\right)\frac{1}{\sqrt{2}}\left(\left|\mathrm{H}\right\rangle - \left|\mathrm{V}\right\rangle\right)\right]\\ &= \frac{1}{\sqrt{2}}\left[\left|\mathrm{HV}\right\rangle - \left|\mathrm{VH}\right\rangle\right] \equiv \left|\Psi^{-}\right\rangle.\end{aligned}$$

[3]This can, for instance, be shown using the Jones matrix formalism. The math is straight forward but somewhat lengthy and not very instructive, so it's not given explicitly here.

In words, rotating the photon polarizations by 45° turns the negatively correlated state $|\Psi^+\rangle$ into the positively correlated state $|\Phi^-\rangle$, whereas $|\Psi^-\rangle$ is preserved. This difference is then measureable for us.

Similarly, $|\Phi^-\rangle$ transforms into $|\Psi^+\rangle$, whereas $|\Phi^+\rangle$ stays intact. Including the π-phase shift we can obtain via the quarter-wave plate, we have all ingredients to create and distinguish all four Bell states.

Summarizing, in practice, this is done as follows: First, the HWPs are adjusted such that the desired kind of correlations are found (positive or negative). This defines the zero-point of the polarization rotation. To adjust the relative phase of the state, we rotate the polarization via both HWPs by 45°. Then, we can adjust the QWPs to make sure this rotation corresponds to proper transformation described above (same or orthogonal correlations compared to before the rotation). After rotating both HWPs back, the state is fully prepared.

A.2 Bell Angles

We speak of a suitable set of four angles for the full measurent of S as 'Bell angles', if they result in a maximum violation of Bell's inequality, $|S| = 2\sqrt{2}$. However, they are not the same for all four Bell states. Figure 3 shows, for fixed angles Θ_A and Θ_B, where, depending on both Θ'_A and Θ'_B, Bell's inequaltiy can be expected to be violated.

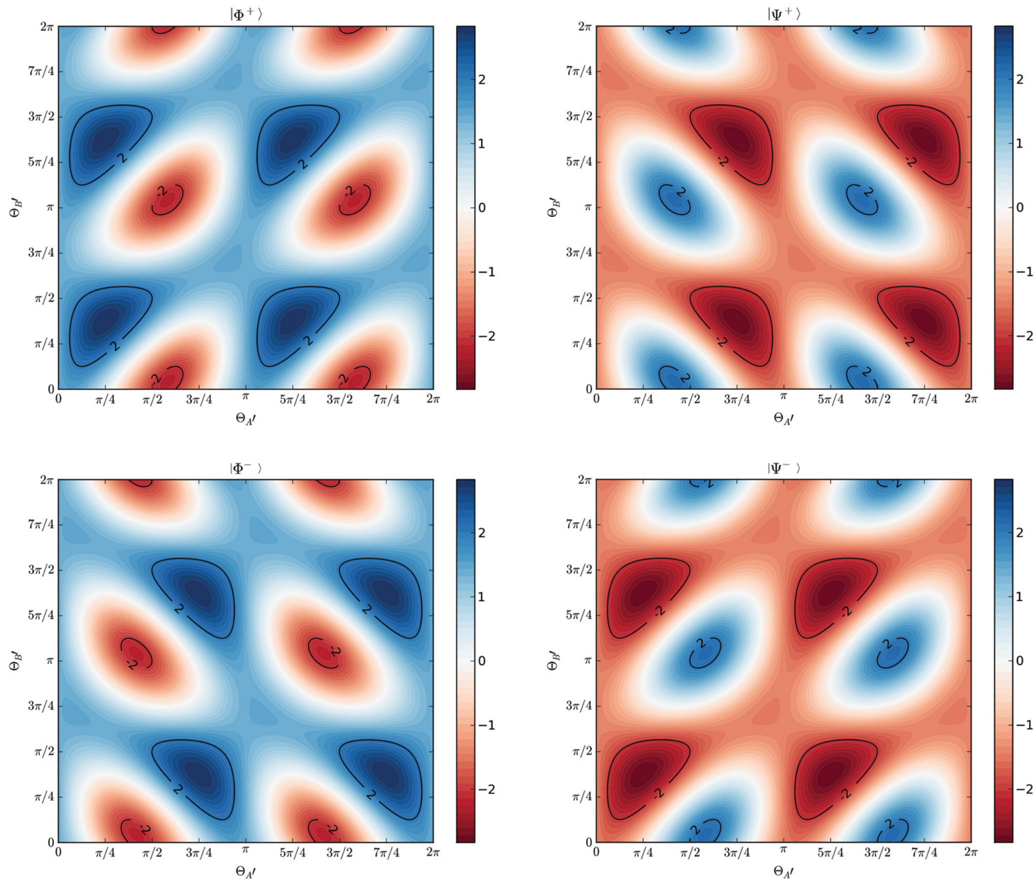

Figure 3: S for all four Bell states. For all plots, $\Theta_A = 0°$ and $\Theta_B = 22.5°$, and both secondary angles are varied from 0 to 2π. The black lines enclose areas where $|S| > 2$.

B Error Analysis

B.1 Counting Error

In the experiment, a count rate is measured, that fluctuates around its mean in time. This means, that the number of counts obey Poissonian distribution, and the standard deviation of the number of counts N (i.e., the error that has to be taken into account during evaluation) is given by $\sigma = \sqrt{N}$.[4]

B.2 Bell State Quality

In an ideal setup that makes use of an ideal EPR pair source, we would expect a maximum value of S of $2\sqrt{2}$ for the four Bell states. However, in practice this might not be the case. If you turn the wave plates in such a fashion that you'd measure $E(0,0)$, you would, assuming negative correlations, expect to obtain $E = 1$. As you will see, this is not the case, the contrast is limited, i.e., you observe counts for the positive correlations as well.

Let us assume that the contrast is limited by a universal factor $\alpha < 1$ such that

$$E_{\text{real}} = \alpha \cdot E_{\text{ideal}},$$

Where we shall assume for now that this is a constant of the EPR source and the setup, without being more specific what causes this limitation. This leads then to

$$S_{\text{real}} = \alpha \cdot S_{\text{ideal}},$$

which will lead to the fact that for maximum violation of Bell's inquality we observe

$$S_{\text{real}} = \alpha \cdot 2\sqrt{2}.$$

B.3 Angular Error

To keep confusion at a reasonable level, let's ignore the error resulting from inaccurate wave plate adjustment for now.

[4]This obviously means, that the larger N is, the smaller the relative error will be. Since we can assume a time measurement to be free of error, this error can be minimized by measuring count rates with respect to a longer time interval.